SIMILARITY SOLUTIONS FOR BOUNDARY-LAYER EQUATIONS WITH INTERACTION

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The classification of flow regimes near small two-dimensional roughness elements on the surface of a body in supersonic viscous flow is given in [1] as the Reynolds number based on the free stream velocity and the characteristic length of the body tends to infinity. The principal similarity parameters that determine the flow characteristics, the type of equations, and the boundary conditions have been established using the well-known method of matched asymptotic expansions. In particular, it has been determined that the roughness with a characteristic range $\varepsilon^{3/2} < b/l < \varepsilon^{3/4}$ and a characteristic transverse dimension $a/l \sim O[\varepsilon(b/l)^{1/3}]$, where $\varepsilon = \mathrm{Re}^{-1/2}$, remain within a subsonic wall layer of the undisturbed boundary layer and the flow near such roughnesses in the first approximation as $\varepsilon \to 0$ is described by Prandtl's incompressible boundary layer equations. The pressure distribution in this case is determined from the condition for the interaction of the roughness with the near wall layer of the undisturbed boundary layer, i.e., for the given flow conditions there is no interaction with the external flow in the first approximation and the change in roughness height is compensated by a change in the displacement thickness of the near wall layer of the undisturbed boundary layer. Hence the flow under study is referred to as "compensating" flow. The formulation of the boundary-value problem for the "compensated" flow over roughness is also described in [2] and the solution in the wake of a finite roughness on a body has been obtained. A complete solution to the nonlinear boundary-value problem is obtained in [3]. The classification of flow conditions near small roughness is given in [4] and the solutions for different flow conditions around the roughness as the local similarity parameters approach their limiting values are given in [5]. The similarity solutions to the "compensating" flow past roughnesses on the surface of a body are investigated in this paper.

1. "Compensating" flow past small roughnesses is described by the following boundary-value problems (see, e.g., [3]):

$$\begin{gathered}\Pi\psi''' = p^{\cdot} + \psi'^{\cdot}\psi' - \psi^{\cdot}\psi'',\ \Pi H''/\mathrm{Pr} = -\psi^{\cdot}H' + H^{\cdot}\psi', \\ \Pi = \mu_w b_1/A\rho_w a_1^3, \quad \psi = \psi' = H = 0 \quad (y = f(x)), \\ \psi'' \to 1,\ H \to \sqrt{2\psi},\ p(x) \to \psi - y^2/2 (y \to \infty),\ f(x) \to 0,\ p(x) \to 0, \\ \psi \to y^2/2,\ H \to y (x \to -\infty),\end{gathered} \tag{1.1}$$

where x, y are the usual Cartesian coordinates; $(\ )^{\cdot}$ and $(\ )'$ refer to differentiation with respect to the longitudinal and the transverse coordinates; $\psi(x, y)$, $H(x, y)$, and $p(x)$ are the stream function, enthalpy fluctuation with respect to its value at the surface, and the pressure, respectively; Pr and $f(x)$ are Prandtl's number and the normalized form of the roughness; μ_w, ρ_w, and A are the values of the dynamic viscosity coefficient, density, and the shear stress at the surface of the body in the undisturbed boundary layer at the point where the small roughness is located; a_1 and b_1 are the transverse and longitudinal dimensions of the roughness. In the chosen system of variables, the shear stress τ and the heat flux q in the undisturbed boundary layer at the body surface are equal to one.

New variables are introduced

$$\begin{gathered}y = c(x)N + f(x),\ p(x) = d(x) - f^2(x)/2, \\ \psi(x, y) = d(x)\varphi(x, N) + c^2(x)N^2/2 + c(x)f(x)N,\end{gathered}$$

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$$H(x,\ y) = d(x)g(x,\ N)/c(x) + c(x)N,$$

in which the boundary-value problem (1.1) takes the following convenient form for analysis

$$\Pi\varphi''' = \frac{c^3 d^{\cdot}}{d} - [c\,(cf)^{\cdot}N + c^2 c^{\cdot} N^2 + cd^{\cdot}\varphi]\,\varphi'' + \Big[c^2 f^{\cdot} + \frac{cf}{d}(cd^{\cdot} - \tag{1.2}$$
$$- c^{\cdot}d) + \frac{c^3 d^{\cdot}}{d}N + (cd^{\cdot} - c^{\cdot}d)\,\varphi'\Big]\varphi' - \frac{c^3 d^{\cdot}}{d}\varphi + c^3\Big[\varphi'^{\cdot}\Big(N + \frac{f}{c} + \frac{d}{c^2}\varphi'\Big) - \varphi^{\cdot}\Big(1 + \frac{d}{c^2}\varphi''\Big)\Big],$$
$$\frac{\Pi}{\Pr}g'' = -\frac{c^4 f^{\cdot}}{d}N - \frac{c^3 d^{\cdot}}{d}\varphi + c^2 c^{\cdot} N\varphi' - c^3\varphi^{\cdot} - [c\,(cf)^{\cdot}N + c^2 c^{\cdot} N^2 + cd^{\cdot}\varphi +$$
$$+ cd\varphi^{\cdot}]\,g' + \Big(\frac{c^3 d^{\cdot}}{d} - c^2 c^{\cdot}\Big)\Big(N + \frac{f}{c} + \frac{d}{c^2}\varphi'\Big)g + c^3 g^{\cdot}\Big(N + \frac{f}{c} + \frac{d}{c^2}\varphi'\Big),$$
$$\varphi(x,\ 0) = 0,\ \varphi'(x,\ 0) = -cf/d,\ \varphi''(x,\ \infty) = 0,\ \varphi(x,\ \infty) = 1,$$
$$g(x,\ 0) = 0,\ g'(x,\ \infty) = 0,\ f(x) \to 0,\ d(x) \to 0\ (x \to -\infty).$$

It is necessary to mention that in the boundary-value problem (1.2) the interaction condition $\varphi(x, \infty) = 1$ does not increase the order of the derivatives with respect to the longitudinal coordinate x present in the boundary-value problem (1.2), the equations remain parabolic, and the upstream propagation of disturbances is not considered in their solution.

Since the roughness interacts only with the subsonic part of the undisturbed boundary layer, the pressure fluctuation is positive when the flow is past a depression on the surface ($f(x) < 0$): $\Delta p > 0$ and $d(x) > f^2(x)/2 > 0$. If, however, the roughness is a protuberance ($f(x) > 0$), then $\Delta p < 0$ and $d(x) < 0$. In what follows the upper sign will always refer to the case of the flow past a protuberance and the lower sign will refer to the case of depression.

The following arbitrary normalized function $c(x) > 0$ is considered:

$$c(x) = D\ [\mp d(x)]^{\gamma}.$$

Obviously, only when $f(x) = \pm[\mp d(x)]^{1-\gamma}$ do the boundary conditions of the given problem (1.2) take the similarity form

$$\varphi(0) = 0,\ \varphi'(0) = D,\ \varphi''(\infty) = 0,\ \varphi(\infty) = 1, \tag{1.3}$$
$$g(0) = 0,\ g'(\infty) = 0.$$

For the existence of a nontrivial self-similar solution of the boundary-value problem (1.2), it is necessary to satisfy the condition

$$c^3(x)d^{\cdot}(x)/d(x) = \beta. \tag{1.4}$$

Integration of the differential equation (1.4) gives (the constant of integration can be made zero by shifting the origin)

$$d(x) = \mp(3\beta\gamma x/D^3)^{1/3\gamma},\ f(x) = \pm(3\beta\gamma x/D^3)^{1/3\gamma - 1/3}, \tag{1.5}$$
$$c(x) = (3\beta\gamma x)^{1/3}.$$

Now it is possible to obtain an estimate for the variation in the fluctuations of the shear stress or the heat flux at the roughness surface with respect to their values in the undisturbed boundary layer on the surface of the body

$$(\tau - 1) \sim (q - 1) \sim d(x)/c^2(x) \sim x^{1/3\gamma - 2/3}. \tag{1.6}$$

It is seen from the relation (1.5) that the product $\beta\gamma x$ should be >0, i.e., for different combinations of the signs of the quantities β, γ, and the coordinate x different similar solutions of the boundary-value problem (1.2) can be obtained.

2. Let initially, $x, \beta, \gamma > 0$, i.e., the downstream propagation of disturbances is being analyzed, and the pressure disturbance in this case increases in absolute value $|d(x)|^{\cdot} > 0$. Such flows are conditionally called flows with compression.

An estimate of the order of magnitude of the terms in the equations of the boundary-value problem (1.2) in powers of x shows that with the use of the relations (1.5) and $\gamma = 1/2$ the Eqs. (1.2) reduce to nonlinear self-similar equations for all $x > 0$:

$$\Pi\varphi''' = \beta\Big[1 - \Big(\pm\frac{N}{D} + \frac{N^2}{2} \mp \frac{\varphi}{D^2}\Big)\varphi'' + \Big(\pm\frac{1}{D} + N \mp \frac{\varphi'}{2D^2}\Big)\varphi' - \varphi\Big], \tag{2.1}$$

$$\frac{\Pi}{\Pr} g'' = \beta\left[\frac{DN}{2} - \varphi + \frac{N\varphi'}{2} - \left(\pm\frac{N}{D} + \frac{N^2}{2} \mp \frac{\varphi}{D^2}\right)g' + \frac{1}{2}\left(\pm\frac{1}{D} + N \mp \frac{\varphi'}{D^2}\right)g\right],$$

whose solution should satisfy the boundary condition (1.3). In terms of the new variables

$$N = (\Pi/\beta)^{1/3} n, \quad g(N) = (\beta/\Pi)^{1/3} G(n)$$

Eq. (2.1) and the boundary conditions (1.3) take the following convenient form for numerical integration:

$$\varphi''' = 1 - \left(\pm\frac{n}{E} + \frac{n^2}{2} \mp \frac{\varphi}{E^2}\right)\varphi'' + \left(\pm\frac{1}{E} + n \mp \frac{\varphi'}{2E^2}\right)\varphi' - \varphi, \tag{2.2}$$

$$\frac{G''}{\Pr} = \frac{En}{2} - \varphi + \frac{n\varphi'}{2} - \left(\pm\frac{n}{E} + \frac{n^2}{2} \mp \frac{\varphi}{E^2}\right)G' + \frac{1}{2}\left(\pm\frac{1}{E} + n \mp \frac{\varphi'}{E^2}\right)G,$$

$$\varphi(0) = 0, \ \varphi'(0) = E, \ \varphi''(\infty) = 0, \ \varphi(\infty) = 1, \ G(0) = 0,$$

$$G'(\infty) = 0, \ E = D(\Pi/\beta)^{1/3}.$$

Numerical integration of the boundary-value problem (2.2) makes it possible to determine the value of the parameter E, find the roughness shape $f(x) = \pm(3\Pi x/2E^3)^{1/3}$, distribution of pressure fluctuations $d(x) = \mp f^2(x)$, magnitude of the shear stress disturbance $\tau - 1 = \mp\varphi''(0)/E^2$, and the heat flux $q - 1 = \mp G'(0)/E^2$:

$$\tau - 1 = 0{,}7350, \ q - 1 = 0{,}1972 \ (f(x) = 0{,}6775(\Pi x)^{1/3}),$$

$$\tau - 1 = -0{,}7182, \ q - 1 = -0{,}2569 \ (f(x) = -0{,}9503(\Pi x)^{1/3})$$

for all $x > 0$. Profiles of the functions $\varphi(n)$, $\varphi'(n)$, and $-G(n)$ for $f(x) > 0$ are shown in Fig. 1 (curves 1-3), with Pr = 0.71 in all the computations. If, however, $\gamma \neq 1/2$, then the boundary-value problem (1.2) will have similarity form only when $x^{1/3}\gamma^{-2/3} \ll 1$, i.e., when $x \gg 1$ for $\gamma > 1/2$ and when $x \ll 1$ for $\gamma < 1/2$:

$$\varphi''' = 1 - \gamma n^2\varphi'' + n\varphi' - \varphi, \tag{2.3}$$

$$G''/\Pr = (1-\gamma)En - \varphi + \gamma n\varphi' - \gamma n^2 G' + (1-\gamma)nG,$$

$$\varphi(0) = 0, \ \varphi'(0) = E, \ \varphi''(\infty) = 0, \ \varphi(\infty) = 1, \ G(0) = 0,$$

$$G'(\infty) = 0.$$

The numerical solution of the boundary-value problem (2.3) has been obtained for a wide range of $\gamma > 0$. Figure 2 shows the dependence of E, $-\varphi''(0)$, and $-G'(0)$ on γ (curves 1-3); the dependence of $f(x)$, $d(x)$, $(\tau - 1)$, and $(q - 1)$ in this case will be determined by the equations

$$f(x) = \pm(3\Pi\gamma x/E^3)^{1/3\gamma - 1/3}, \ d(x) = \mp(3\Pi\gamma x/E^3)^{1/3\gamma},$$

$$\tau - 1 = \mp[\varphi''(0)/E^2](3\Pi\gamma x/E^3)^{1/3\gamma - 2/3}, \ q - 1 = \mp[G'(0)/E^2](3\Pi\gamma x/E^3)^{1/3\gamma - 2/3}.$$

It is important to observe from these that for $x \gg 1$ the pressure disturbances amplify $1 < |d(x)| < x^{2/3}$ for the flow past the roughness ($x^{-1/3} < |f(x)| < x^{1/3}$), and the fluctuations of the shear stress (or heat flux) decrease $x^{-2/3} < |\tau - 1| < 1$. In particular, for the flow past a protuberance $f(x) = \pm 1)d(x) \sim x^{1/3}$ and $(\tau - 1) \sim (q - 1) \sim x^{-1/3}$, i.e., the fluctuations of shear stress and heat flux decrease very weakly with increase in x.

3. Let $x > 0$, but β, $\gamma < 0$, i.e., pressure fluctuations decrease downstream in absolute value $|d(x)|^{\cdot} < 0$. Such flows are conditionally called flows with rarefaction. In this case also the relations (1.5) and (1.6) are true, if the signs of β and γ in them are replaced by their opposite:

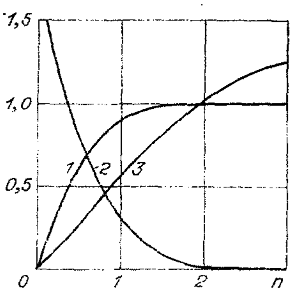

Fig. 1.

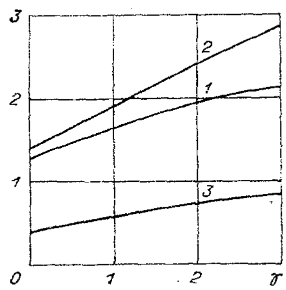

Fig. 2.

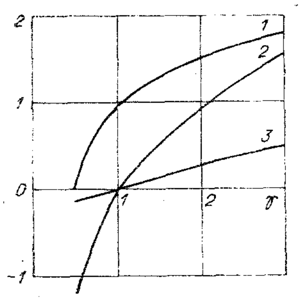

Fig. 3.

$$d(x) = \mp(3\beta\gamma x/D^3)^{-1/3\gamma},\ f(x) = \pm(3\beta\gamma x/D^3)^{-1/3\gamma-1/3},$$
$$c(x) = (3\beta\gamma x)^{1/3},\ (\tau - 1) \sim (q - 1) \sim x^{-1/3\gamma-2/3}$$

and further assume $\beta,\ \gamma > 0$. It is possible to observe that in this case the original boundary-value problem (1.2) takes the similarity form only when $x \gg 1$:

$$\begin{gathered}\varphi''' = -1 - \gamma n^2\varphi'' - n\varphi' + \varphi,\\ G''/\mathrm{Pr} = -(1+\gamma)En + \varphi + \gamma n\varphi' - \gamma n^2 G' - (1+\gamma)nG,\\ \varphi(0) = 0,\ \varphi'(0) = E,\ \varphi''(\infty) = 0,\ \varphi(\infty) = 1,\ G(0) = 0,\\ G'(\infty) = 0.\end{gathered} \tag{3.1}$$

The numerical solution of the boundary-value problem (3.1) has been obtained for a wide range of $\gamma > 0$, the dependence of E, $-\varphi''(0)$, and $-G'(0)$ on γ are shown in Fig. 3 (curves 1-3). The dependence of f(x), d(x), $(\tau - 1)$, and $(q - 1)$ in this case will be determined by the equations

$$f(x) = \pm(3\Pi\gamma x/E^3)^{-1/3\gamma-1/3},\ d(x) = \mp(3\Pi\gamma x/E^3)^{-1/3\gamma},$$
$$\tau - 1 = \mp[\varphi''(0)/E^2](3\Pi\gamma x/E^3)^{-1/3\gamma-2/3},\ q - 1 = \mp[G'(0)/E^2](3\Pi\gamma x/E^3)^{-1/3\gamma-2/3},$$

from which it is seen that for the flows with rarefaction all disturbances damp out with increase in the longitudinal coordinate x for $\gamma > 0$. Results of computations show that for $\gamma = 1$ $\varphi''(0) = G'(0) = 0$ and, that means,

$$d(x) \sim x^{-1/3},\ \tau = q = 1\ (f(x) \sim x^{-2/3}).$$

Besides, it happens that when $\gamma = 1/2$, $E = 0$. It follows that the solution of the boundary-value problem (3.1) has a meaning only when $\gamma \geqslant 1/2$ when $E \geqslant 0$, since $E = 0$ corresponds to the limiting case, viz., the flow in the wake of a finite roughness on the surface of a body, i.e., for $f(x) \equiv 0$ when $x \gg 1$ (in this case it is necessary to put $f(x) = f^{\cdot}(x) \equiv 0$ in all the equations and then use the relation $D = 1$ and $E = (\Pi/\beta)^{1/3}$). Precisely such a flow was examined in [2] and here the disturbances dampen with the increase in the longitudinal coordinate x in the following manner:

$$d(x) \sim x^{-2/3},\ (\tau - 1) \sim (q - 1) \sim x^{-4/3}.$$

This also means that for the flow past roughnesses $f(x) \lesssim x^{-1}$ for $x \gg 1$ the damping of disturbances is already determined by the interaction of the near wall layer of the undisturbed boundary layer with the surface of the body $f(x) \equiv 0$, and not with the roughness itself.

When $\gamma = 1$ the solution of the boundary-value problem (3.1) for the function $\varphi(n)$ can be obtained in an explicit form

$$\varphi''(n) = \varphi''(0)\left(1 - n\exp\left(-\frac{n^3}{3}\right)\int_0^n \xi\exp\left(\frac{\xi^3}{3}\right)d\xi\right) - n\exp\left(-\frac{n^3}{3}\right). \tag{3.2}$$

However, this solution is not unique, since $\varphi''(0)$ remains undetermined. If $\varphi''(0) \neq 0$, then it follows from an analysis of the Eq. (3.2) that $\varphi''(n) \sim \varphi''(0)/n^3\ (n \to \infty)$.

On the other hand, results of computations show that in the neighborhood of the point $\gamma = 1$, $\varphi''(0)$ changes its sign (see Fig. 3), and hence it is possible to consider that there is a solution in which $\varphi''(0) = 0$ for $\gamma = 1$. Then the fluctuations of shear stresses damp out exponentially as $n \to \infty$ and the quantity $E = 3^{-1/3}\Gamma(2/3) = 0.93889$ (results of computations give $E = 0.93887$).

Specific analytical results can be obtained with the Fourier transform in the x variable for the linearized boundary-value problem (1.2) for $\varphi(x, N)$. After a few simple computations (see, e.g., [2]) the following relation between E and γ is obtained:

$$E = (3\gamma)^{1/3}\Gamma(\pm 1/3\gamma + 1)\theta^{-4/3}\Gamma^{-1}(\pm 1/3\gamma + 2/3), \tag{3.3}$$

where $\theta = [-3\mathrm{Ai}'(0)]^{3/4} \approx 0.8272$; $\gamma \geqslant 0$; but here the sign + corresponds to the solution of the boundary-value problem (2.3) and the sign − refers to the boundary-value problem (3.1). The values of E computed from this equation practically coincide with the results shown in Fig. 2, 3. In particular, for the flow with rarefaction it follows from (3.3) that E = 0 when $\gamma = 1/2$.

Let $x < 0$, then, it follows from (1.5) that the quantities β and γ should have opposite signs. Since the roughness height $f(x) \to 0$ as $x \to -\infty$, then $|d(x)|^{\cdot} > 0$, and, consequently, $\beta > 0$ and $\gamma < 0$.

Here, as in Sec. 3, the original boundary-value problem (1.2) can take the similarity form only for $-x \gg 1$:

$$\varphi''' = -1 + \gamma n^2\varphi'' - n\varphi' + \varphi, \ \varphi(0) = 0, \ \varphi'(0) = E, \quad \varphi''(\infty) = 0, \ \varphi(\infty) = 1. \tag{4.1}$$

It is necessary to observe that this boundary-value problem differs from (3.1) only by the sign of the term $\gamma n^2\varphi''$.

For the function $z(n) = \varphi''(n)\exp(-\gamma n^3/6)$, as $n \to \infty$, it is possible to obtain the equation $z'' - \gamma^2 n^4 z/4 = 0$, whose solution is expressed in terms of the modified Bessel function [6]. Considering, that as $n \to \infty$ the function $z(n)$ should decrease, it is possible to get

$$z(n) \sim n^{1/2}K_{1/6}(\gamma n^3/6).$$

Then it follows from the asymptotic expression for the function $K_{1/6}(\gamma n^3/6)$, that $\varphi''(n) \to$ constant as $n \to \infty$. The boundary condition $\varphi''(\infty) = 0$ then gives the solution of the type $\varphi''(n) \equiv 0$, which does not satisfy the remaining boundary conditions of the problem (4.1), and, it means that the boundary-value problem (4.1) does not have a solution. This shows that the original boundary-value problem (1.2) does not have a similarity solution for the roughness extending to infinity.

LITERATURE CITED

1. V. V. Bogolepov and V. Ya. Neiland, "Flow past small roughness on the surface of a body in supersonic viscous flow," Tr. TsAGI, No. 1363 (1971).
2. J. C. R. Hunt, "A theory for the laminar wake of two-dimensional body in boundary layer," J. Fluid Mech., 49, Pt. 1 (1971).
3. V. V. Bogolepov, "Computation of the supersonic boundary-layer interaction with an obstacle," Uch. Zap. TsAGI, 5, No. 6 (1974).
4. V. V. Bogolepov and V. Ya. Neiland, "Investigation of the local disturbances of viscous supersonic flows," in: Aerodynamics [in Russian], Nauka, Moscow (1976).
5. V. V. Bogolepov, "Investigation of the limiting solutions for the supersonic viscous flow past small roughness on the surface of a body," Trudy TsAGI, No. 1812 (1977).
6. M. Abramovits and I. Stigan (eds.), Handbook on Special Functions [in Russian], Nauka, Moscow (1979).